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PERTURBATION THEORY APPLIED TO
ORDINARY DIFFERENTIAL OPERATORS

by

DOUGLAS W. PITNEY

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GRADUATE STUDIES IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE
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ON PERTURBATION THEORY APPLIED TO
ORDINARY DIFFERENTIAL OPERATORS

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DOUGLAS W. PITNEY

A THESIS
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ABSTRACT

This thesis is devoted to the study of standard perturbation techniques applied to two well known second order differential operators. In the introductory chapter we present the definitions and basic theorems that are necessary to develop these techniques. Chapter two deals with Legendre's equation which is perturbed with two unbounded operators. In the third chapter we study Bessel's transformed equation perturbed in the standard way. One example shows the validity of these techniques for any bounded perturbing operator and the second example is a differential perturbing operator. The last chapter deals with some perturbing operators that produced no interesting results.

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CHAPTER I

INTRODUCTION

Perturbation techniques have been used on ordinary differential equations by Bellman [1], Rellich [1], and Cole [2]. In this thesis we use perturbation techniques to look for series solutions to perturbed forms of Legendre's equation and Bessel's equation. Some particular examples of a perturbing operator are investigated. In the last chapter, we indicate some cases for which no results were obtained.

1. PRELIMINARY REMARKS.

The vector space of real valued functions that are continuous for all $x \in [a,b]$ is denoted by $\mathcal{C}[a,b]$ and the space of real valued functions that are twice continuously differentiable for all $x \in [a,b]$ is denoted by $\mathcal{C}^2[a,b]$. Finally, real valued functions that are continuous at all but a finite number of points of an interval and at each point of discontinuity, both the left and right limits of the functions exist, are called piecewise continuous real valued functions. We denote this vector space by $\mathcal{PC}[a,b]$.

Let the real vector space V be defined such that for any $f, g \in V$, the inner product

$$(f, g) = \int_a^b f(x)g(x)dx \quad (1.1)$$

exists for $x \in [a,b]$. V is called a Euclidean or inner product space. The vector space $C^2[a,b]$ with this inner product is denoted by $C^2[a,b]$. The space $P[a,b]$, again with (1.1) as inner product, is the Euclidean space $PC[a,b]$ and $PC[a,b]$ contains $C^2[a,b]$ as a subspace.

For any $f \in V$, the non-negative real number

$$\|f\| = \sqrt{(f,f)} \quad (1.2)$$

is the norm of f . A sequence f_1, f_2, \dots of real valued functions of V is said to be orthogonal if $(f_i, f_j) = 0$ whenever $i \neq j$ and $\|f_j\| \neq 0$ for every j . It is orthonormal if it is orthogonal and $\|f_j\| = 1$ for every j . As we shall see later, Legendre polynomials and Bessel functions form orthogonal sequences which are easily normalized.

Let the sequence $\{f_i\}$ for $f_i \in V$, $i = 1, 2, \dots$, be an orthonormal sequence of functions. Also let $f \in V$. The series

$$\sum_{i=1}^{\infty} (f, f_i) f_i \quad (1.3)$$

is said to be the orthogonal development of f in V with respect to the f_i 's. We say that the sequence $\{f_i\}$, $i = 1, 2, \dots$, is complete if for all $f \in V$, the orthogonal development of f converges to f . This will be shown to be the case for sequences of Legendre polynomials and Bessel functions in $PC[-1,1]$ and $PC[0,1]$ respectively [10].

We are concerned with Legendre's differential equation (Chapter II) and the transformed Bessel's differential equation (Chapter III). In each case we have an equation of the form

$$Ly = \lambda y \quad (1.4)$$

where the differential operator L is defined by

$$Ly = (p(x)y')' + q(x)y, \quad x \in [a,b] \quad (1.5)$$

An operator of the form (1.5) is said to be self-adjoint. The necessary and sufficient condition for the general second order differential operator

$$Dy = fy'' + gy' + hy \quad (1.6)$$

(in which f , g , and h are functions of x) to be self-adjoint is that $g = f'$ [11]. This is clearly the case for Legendre's differential operator where $g = f' = -2x$ and for Bessel's differential operator where $g = f' = 1$.

The values of λ which yield nontrivial solutions of (1.4) are called eigenvalues and the associated solutions are called eigenfunctions or eigenvectors. If L is a linear self-adjoint operator then all of the eigenvalues are real [10].

The set of eigenvalues is called the spectrum [7]. Titchmarsh [16, Ch. 5] gives some criteria that enable us to decide when the problem (1.4) has a discrete or continuous spectrum. The most general conditions for discreteness or continuity are unknown. In this thesis we assume, as is the case for Legendre's equation and Bessel's equation, a discrete spectrum which is bounded above.

Dunford and Schwarz [6, p. 1478] prove that a second order self-

adjoint operator with a discrete spectrum that is bounded above or below has non-degenerate eigenvalues. Therefore, we can disregard the problem of degeneracy.

Let S be a subspace of $C^2[a,b]$ determined by a pair of boundary conditions such that

$$p(x)[y_1(x)y_2'(x) - y_2(x)y_1'(x)] \Big|_a^b = 0$$

for every $y_1, y_2 \in S$. Then any set of eigenvectors belonging to distinct eigenvalues for the problem

$$Ly = \lambda y$$

is orthogonal in $C[a,b]$ with the inner product

$$(y_1, y_2) = \int_a^b y_1(x)y_2(x)dx.$$

Also

$$(Ly_1, y_2) = (y_1, Ly_2)$$

for y_1, y_2 vectors of a Euclidean space [10].

2. THE PERTURBATION PROBLEM.

The method of perturbation [1], [2], [7] is applicable if the operator to be analyzed is, in one sense or another, near to an operator whose spectrum is known. The spectrum of an operator is sensitive even to the slightest changes in the operator.

The operator with a known spectrum, the "unperturbed" operator, will be denoted by L , the "perturbing" operator by B , and the "perturbed" operator depending on a parameter ϵ is denoted by $L + \epsilon B$.

We assume, as is the case for Legendre's differential operator and Bessel's differential operator, that the unperturbed operator L , given by (1.5), is self-adjoint and has discrete isolated eigenvalues λ_{oi} with corresponding normalized eigenvectors y_{oi} .

Suppose the perturbed operator $L + \epsilon B$ has an eigenvalue λ_ϵ which, together with an eigenvector y_ϵ , depends analytically on ϵ . Then expansions

$$\lambda_\epsilon = \lambda_0 + \lambda_1 \epsilon + \lambda_2 \epsilon^2 + \dots, \quad (1.8)$$

$$y_\epsilon = y_0 + y_1 \epsilon + y_2 \epsilon^2 + \dots, \quad (1.9)$$

may be valid for sufficiently small values of ϵ and may be formally substituted into

$$(L + \epsilon B)y = \lambda_\epsilon y_\epsilon. \quad (1.10)$$

We equate coefficients of like powers of ϵ . In the case that the eigenvalues are distinct, if other orders of ϵ are used, the corresponding y_j would be identically zero. The following sequence of equations is obtained:

$$\begin{aligned} (L - \lambda_0)y_0 &= 0, \\ (L - \lambda_0)y_1 &= \lambda_1 y_0 - B y_0, \\ (L - \lambda_0)y_2 &= \lambda_1 y_1 + \lambda_2 y_0 - B y_1, \\ &\dots \end{aligned} \quad (1.11)$$

Of course, the vector y_ε is not uniquely determined by these equations since it may be multiplied by any function of ε . Since $y_0 \neq 0$ this factor could be so chosen that the norm $\|y_\varepsilon\|$ of y_ε is 1 for sufficiently small ε [15]. Instead, we impose the condition

$$(y_0, y_\varepsilon) = 1$$

which implies [7]

$$(y_0, y_1) = (y_0, y_2) = \dots = 0 \quad . \quad (1.12)$$

Now we can establish the following theorem.

THEOREM 1.1. Assuming a point spectrum of non-degenerate eigenvalues, the right member of each equation in (1.11) is orthogonal to y_0 .

PROOF: We take the inner product of the unperturbed equation with y_i , $i = 1, 2, \dots$. Since $L - \lambda_0$ is a linear self-adjoint operator,

$$((L - \lambda_0)y_0, y_i) = (y_0, (L - \lambda_0)y_i) = 0, \quad i = 1, 2, \dots$$

Therefore the left hand side of the system (1.11) is orthogonal to y_0 which yields the desired result.

If λ_0 is an isolated eigenvalue of L and the domain of $L - \lambda_0$ is the space spanned by the ortho-complement set y_1, y_2, \dots of y_0 then $(L - \lambda_0)^{-1}$ exists. Therefore the sequence of equations (1.11) can be solved for y_0, y_1, \dots (the first being satisfied by definition) and y_1, y_2, \dots are made unique by (1.12). Since the sequences $\{f_n\}$, $n = 0, 1, \dots$, that we deal with in Chapters II and III are complete [10]

the solutions y_1, y_2, \dots are given by

$$y_i = \sum_{n=0}^{\infty} (y_i, f_n) f_n, \quad i = 1, 2, \dots \quad (1.13)$$

Theorem (1.1) and the equality (1.12) yield the values

$$\lambda_1 = (By_0, y_0), \quad \lambda_2 = (By_1, y_0), \dots \quad (1.14)$$

of the expansion coefficients of the eigenvalue λ_ε .

A linear operator L , which assigns to each vector y in V a vector Ly in V , is called "bounded" if the inequality $|Ly| \leq a|y|$ holds for all vectors y with an appropriate number a . If B is a linear self-adjoint bounded operator in a neighbourhood of $\varepsilon = 0$, the convergence of (1.8) and (1.9) is guaranteed by Rellich [12, p. 57]. We study a bounded operator in Chapter III. However, if this is not the case, although the formal power series may always be written down, the convergence of these series must be demonstrated explicitly in each case, as we will do with examples in Chapters II and III.

CHAPTER II

LEGENDRE'S DIFFERENTIAL EQUATION

In this chapter, we study Legendre's differential equation [14, p. 76] using the perturbation techniques of Chapter I. Some interesting results are obtained, particularly when specific examples of the perturbing operator B are used. When a simple first order differential operator is chosen, it is found that only the eigenvectors have nonzero perturbing terms. However, when a situation similar to Legendre's associated equation [14, p. 297] is examined, we find that both the eigenvectors and the eigenvalues have nonzero perturbing terms.

1. LEGENDRE POLYNOMIALS.

The Legendre polynomials may be defined by Rodrigues' formula [3, p. 201]:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \quad n = 0, 1, 2, \dots \quad (2.1)$$

It is clear that for each fixed n , $P_n(x)$ is a polynomial of degree n , $P_n(x)$ contains only odd powers of x when n is odd and only even powers when n is even.

Lemma 2.1. $P_n(1) = 1$, $P_n(-1) = (-1)^n$ for all n .

PROOF: The equality $P_n(1) = 1$ obviously holds for P_0 and P_1 . Moreover, if we assume that $P_0(1) = \dots = P_n(1)$, $n \geq 1$, the recurrence relation [10, p. 416]

$$P_{n+1} = \left(\frac{2n+1}{n+1}\right) P_n - \left(\frac{n}{n+1}\right) P_{n-1} \quad (2.2)$$

gives

$$P_{n+1}(1) = \frac{2n+1}{n+1} - \frac{n}{n+1} = 1.$$

The desired result follows by mathematical induction. In much the same way it can be shown that $P_n(-1) = (-1)^n$.

We frequently need to compute integrals of the form

$$\int_{-1}^1 f(x) P_n(x) dx.$$

If f and its first n derivatives are continuous throughout the interval $-1 \leq x \leq 1$, we use integration by parts n times. Since the first $n-1$ derivatives of $(x^2-1)^n$ vanish at the endpoints $x = \pm 1$, we have

$$\begin{aligned} \int_{-1}^1 f(x) P_n(x) dx &= \frac{1}{2^n n!} \int_{-1}^1 f(x) \frac{d^n}{dx^n} (x^2-1)^n dx = \\ &= \frac{(-1)^n}{2^n n!} \int_{-1}^1 (x^2-1)^n \frac{d^n f(x)}{dx^n} dx, \end{aligned}$$

provided that $\frac{d^n f}{dx^n}$ exists and is continuous throughout the interval. Hence we have the theorem:

THEOREM 2.1. If f and its first n derivatives are continuous throughout the interval $-1 \leq x \leq 1$, then

$$\int_{-1}^1 f(x) P_n(x) dx = \frac{(-1)^n}{2^n n!} \int_{-1}^1 (x^2 - 1)^n \frac{d^n}{dx^n} f(x) dx . \quad (2.3)$$

In particular, if $f(x) = x^m$, where m is an integer $0 \leq m < n$, we have

$$\int_{-1}^1 x^m P_n(x) dx = 0 , \quad m = 0, 1, \dots, n-1 . \quad (2.4)$$

2. LEGENDRE'S DIFFERENTIAL EQUATION.

The boundary conditions for Legendre's differential equation

$$Ly_0 - \lambda_0 y_0 = 0 , \quad -1 \leq x \leq 1 , \quad (' = \frac{d}{dx})$$

where $Ly_0 = ((1-x^2)y_0')'$, are simply: the solutions remain finite at $x = \pm 1$. When $\lambda_0 = -n(n+1)$, $n = 0, 1, 2, \dots$, the Legendre polynomials, $P_n(x)$, are the only bounded solutions of (2.5).

It is easy to prove [3, p. 202] that the sequence of functions $\{P_n(x)\}$, $n = 0, 1, 2, \dots$, is an orthogonal sequence for $x \in [-1, 1]$.

THEOREM 2.2. For non-negative integral values of n and m , $x \in [-1, 1]$ and the inner product

$$(P_n, P_m) = \int_{-1}^1 P_n(x) P_m(x) dx , \quad (2.6)$$

we have

$$(P_n, P_m) = \begin{cases} 0 & \text{for } n \neq m, \\ \frac{2}{2n+1} & \text{for } n = m. \end{cases} \quad (2.7)$$

Using familiar normalization techniques

$$(y_o, y_o) = \int_{-1}^1 y_o^2(x) dx = 1 \quad (2.8)$$

if and only if

$$y_o(x) = \sqrt{\frac{2n+1}{2}} P_n(x), \quad n = 0, 1, 2, \dots$$

Since the eigenvalues and eigenvectors of (2.5) depend on n , we shall write

$$\lambda_o = \lambda_{on} = -n(n+1) \quad (2.9)$$

and

$$y_o = y_{on} = \sqrt{\frac{2n+1}{2}} P_n, \quad n = 0, 1, 2, \dots \quad (2.10)$$

The values $\lambda_{1n}, \lambda_{2n}, \dots$ of the expansion coefficients of the eigenvalue λ_e are given by (1.13). Taking the inner product of the second equation in (1.11) with y_{oi} , $i \neq n$, we get

$$(y_{1n}, y_{oi}) = \frac{(By_{on}, y_{oi})}{\lambda_{on} - \lambda_{oi}}. \quad (2.11)$$

A series of the form

$$\sum_{i=0}^{\infty} (f, y_{oi}) y_{oi} \quad (2.12)$$

for an arbitrary function f in $PC[-1,1]$, is known as the Legendre series expansion of the function f . The proof of the following theorem is on page 423 of Kreider [10].

THEOREM 2.3. The Legendre series (2.12) for a piecewise smooth function f in $PC[-1,1]$ converges pointwise everywhere in the (open) interval $(-1,1)$ and has the value $[f(x^+) + f(x^-)]/2$ at each point in the interval. Moreover, the convergence is uniform on any closed subinterval of $(-1,1)$ in which f is continuous.

COROLLARY 2.1. The sequence of Legendre polynomials $\{P_n(x)\}$, $n = 0, 1, \dots$, is complete in $C^2[-1,1]$ with respect to the inner product

$$(P_n, P_m) = \int_{-1}^1 P_n(x) P_m(x) dx.$$

That is, if f is a real valued, twice continuously differentiable function for all x in the interval $[-1,1]$ then $\sum_{n=0}^{\infty} a_n y_{on}(x)$ converges uniformly to $f(x)$ where

$$a_n = (f, y_{on}) = \int_{-1}^1 f(x) y_{on}(x) dx, \quad y_{on} = \sqrt{\frac{2n+1}{2}} P_n. \quad (2.13)$$

Now by (1.12), $(y_{kn}, y_{on}) = 0$ for all $k = 1, 2, \dots$ so

$$y_{kn} = \sum_{\substack{i=0 \\ i \neq n}}^{\infty} (y_{kn}, y_{oi}) y_{oi} \quad (2.14)$$

where

$$(y_{kn}, y_{oi}) = \frac{1}{\lambda_{on} - \lambda_{oi}} [(By_{k-1}, y_{oi}) - \lambda_{k-1,n}(y_{1n}, y_{oi}) - \dots - \lambda_{1n}(y_{k-1,n}, y_{oi})] \quad (2.15)$$

3. AN EXAMPLE - THE EIGENVECTORS PERTURBED.

Consider the operator

$$B = \frac{d}{dx} \quad ,$$

for x in the interval $-1 \leq x \leq 1$. Clearly y' , ($' = \frac{d}{dx}$), is not an eigenvector of Legendre's equation and the perturbed equation (1.10) can be written

$$((1-x^2)y')' + \epsilon y' = \lambda y \quad .$$

From (1.13),

$$\lambda_{1n} = (y'_{on}, y_{on}) = \int_{-1}^1 y'_{on}(x) y_{on}(x) dx = 0 \quad (2.16)$$

and from (2.14),

$$\begin{aligned}
y_{1n}(x) &= \sum_{\substack{i=0 \\ i \neq n}}^{\infty} (y_{1n}, y_{oi}) y_{oi} \\
&= \sum_{\substack{i=0 \\ i \neq n}}^{\infty} \frac{\sqrt{\frac{2n+1}{2}} \left(\frac{2i+1}{2}\right) P_i(x)}{(\lambda_{on} - \lambda_{oi})} \int_{-1}^1 P'_n(x) P_i(x) dx .
\end{aligned}$$

THEOREM 2.4. For non-negative integral values of n and i ,

$$\int_{-1}^1 P'_n(x) P_i(x) dx = \begin{cases} 2 & \text{if } i < n \text{ and } n+i \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \quad (2.17)$$

PROOF: If $i > n$, the result follows immediately from Theorem 2.1.

If $n+i$ is even, $P'_n(x) P_i(x)$ is an odd function and the integral is zero. Now if $i < n$ and $n+i$ is odd, we integrate by parts.

$$\int_{-1}^1 P'_n(x) P_i(x) dx = P_n(x) P_i(x) \Big|_{-1}^1 - \int_{-1}^1 P'_i(x) P_n(x) dx .$$

By Theorem 2.1, $\int_{-1}^1 P'_i(x) P_n(x) dx$ is zero because $i < n$. Therefore from Lemma 2.1,

$$\int_{-1}^1 P'_n(x) P_i(x) dx = 1 - (-1)^{i+n} = \begin{cases} 2 & \text{if } i+n \text{ is odd,} \\ 0 & \text{if } i+n \text{ is even.} \end{cases}$$

This completes the proof of the theorem.

This means

$$y_{1n}(x) = \sqrt{\frac{2n+1}{2}} \sum_{\substack{i=0 \\ n+i \text{ odd}}}^{\infty} \frac{2i+1}{\lambda_{on} - \lambda_{oi}} P_i(x) .$$

Now,

$$\begin{aligned} \lambda_{2n} = (By_{1n}, y_{on}) &= \int_{-1}^1 y'_{1n}(x) y_{on}(x) dx = \\ &= \frac{2n+1}{2} \sum_{\substack{i=0 \\ n+i \text{ odd}}}^{\infty} \frac{2i+1}{\lambda_{on} - \lambda_{oi}} \int_{-1}^1 P'_i P_n dx . \end{aligned}$$

But $i < n+1$, so $\lambda_{2n} = 0$ by Theorem 2.1.

Simplifying the third perturbed eigenvector we have

$$y_{2n}(x) = \sqrt{\frac{2n+1}{2}} \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \sum_{\substack{i=0 \\ n+i \text{ odd}}}^{\infty} \frac{(2k+1)(2i+1) P_k(x)}{2(\lambda_{on} - \lambda_{ok})(\lambda_{on} - \lambda_{oi})} \int_{-1}^1 P'_i(x) P_k(x) dx .$$

From Theorem 2.4,

$$\int_{-1}^1 P'_i(x) P_k(x) dx = \begin{cases} 2 & \text{if } i > k \text{ and } i+k \text{ odd} , \\ 0 , & \text{otherwise} . \end{cases} \quad (2.18)$$

Therefore,

$$y_{2n}(x) = \sqrt{\frac{2n+1}{2}} \sum_{\substack{k=0 \\ k+n \text{ even}}}^{n-2} \frac{2k+1}{\lambda_{on} - \lambda_{ok}} \left\{ \sum_{\substack{i=k+1 \\ k+i \text{ odd}}}^{n-1} \frac{2i+1}{\lambda_{on} - \lambda_{oi}} \right\} P_k(x) . \quad (2.19)$$

Clearly, each successive eigenvalue will contain an expression of the form:

$$\int_{-1}^1 P'_j(x) P_n(x) dx$$

which is zero because $j < n$ in each case. Therefore,

$$\lambda_{\varepsilon} = \lambda_{on} = -n(n+1), \quad n = 0, 1, \dots$$

For $k = 3$ in (2.14) and (2.15), we get

$$\begin{aligned} y_{3n} &= \sum_{\substack{\ell=0 \\ \ell \neq n}}^{\infty} (y_{3n, y_{o\ell}}) y_{o\ell} = \sum_{\substack{\ell=0 \\ \ell \neq n}}^{\infty} \frac{y_{o\ell}}{\lambda_{on} - \lambda_{o\ell}} \{ (By_{2n, y_{o\ell}})^{-\lambda_{2n}} (y_{1n, y_{o\ell}})^{-\lambda_{1n}} (y_{2n, y_{o\ell}}) \} \\ &= \sum_{\substack{\ell=0 \\ \ell \neq n}}^{\infty} \frac{y_{o\ell}}{\lambda_{on} - \lambda_{o\ell}} (y'_{2n, y_{o\ell}}) . \end{aligned}$$

Substituting for y_{2n} and $y_{o\ell}$ we get

$$y_{3n} = \sqrt{\frac{2n+1}{2}} \sum_{\substack{\ell=0 \\ \ell \neq n}}^{\infty} \frac{P_{\ell}(\frac{2\ell+1}{2})}{\lambda_{on} - \lambda_{o\ell}} \left[\sum_{\substack{k=0 \\ k \neq n}}^{n-2} \frac{2k+1}{\lambda_{on} - \lambda_{ok}} \sum_{\substack{i=k+1 \\ k+i \text{ odd}}}^{n-1} \frac{2i+1}{\lambda_{on} - \lambda_{oi}} \right] \int_{-1}^1 P'_k(x) P_{\ell}(x) dx .$$

From Theorem 2.4,

$$y_{3n} = \sqrt{\frac{2n+1}{2}} \sum_{\substack{\ell=0 \\ n+\ell \text{ odd}}}^{n-3} \frac{2\ell+1}{\lambda_{on} - \lambda_{o\ell}} \left[\sum_{\substack{k=\ell+1 \\ k+n \text{ even}}}^{n-2} \frac{2k+1}{\lambda_{on} - \lambda_{ok}} \left(\sum_{\substack{i=k+1 \\ k+i \text{ odd}}}^{n-1} \frac{2i+1}{\lambda_{on} - \lambda_{oi}} \right) \right] P_{\ell}(x) .$$

(2.20)

Clearly, the series

$$y_{\varepsilon} = \sum_{i=0}^{\infty} u_{in}(x) \varepsilon^i$$

becomes the finite series

$$y_{\varepsilon} = \sum_{i=0}^{n-1} y_{in}(x) \varepsilon^i$$

which converges for all ε .

4. PERTURBING LEGENDRE'S ASSOCIATED EQUATION.

The equation

$$((1-x^2)y')' + [n(n+1)-m^2/(1-x^2)]y = 0, \quad x \neq \pm 1, \quad (2.21)$$

which would be Legendre's equation were it not for the term $m^2/(1-x^2)$, $x \neq \pm 1$, is called Legendre's associated equation [18, p. 195]. There exist solutions

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \quad (2.22)$$

for integral values of n and m which are called the associated Legendre functions. If $m = 0$, (2.21) and (2.22) immediately reduce to Legendre's differential equation and Legendre polynomials, respectively. We shall consider (2.21) as a perturbed form of Legendre's equation by taking $m = \varepsilon$ to be a small non-negative real number.

We use ε^2 in the numerator of the perturbing operator to avoid exponents of the form $\sqrt{\varepsilon}$. Now the perturbed equation can be

written

$$(1-x^2)y'' - 2xy' - y - \frac{\varepsilon^2}{1-x^2}y = 0, \quad x \neq \pm 1. \quad (2.23)$$

In order to remove the denominator of

$$\frac{\varepsilon^2}{1-x^2}y,$$

we try a solution of the form

$$y = (1-x^2)^\alpha h(x)$$

where α is a constant. Differentiating twice, we have

$$y' = -2\alpha x(1-x^2)^{\alpha-1}h(x) + (1-x^2)^\alpha h'(x)$$

and

$$\begin{aligned} y'' = & -2\alpha(1-x^2)^{\alpha-1}h(x) + 4\alpha(\alpha-1)x^2(1-x^2)^{\alpha-2}h(x) - \\ & - 4\alpha x(1-x^2)^{\alpha-1}h'(x) + (1-x^2)^\alpha h''(x). \end{aligned}$$

Substituting into (2.23) and simplifying, we get

$$\begin{aligned} (1-x^2)^{\alpha+1}h''(x) - 2(2\alpha+1)x(1-x^2)^\alpha h'(x) + \\ + [(4\alpha^2 - \varepsilon^2)x^2(1-x^2)^{\alpha-1} - (2\alpha+\lambda)(1-x^2)^\alpha]h(x) = 0 \end{aligned}$$

which can be written

$$\begin{aligned} [(4\alpha^2 - \varepsilon^2)x^2(1-x^2)^{\alpha-1} - (\lambda+2\alpha+\varepsilon^2)(1-x^2)^\alpha]h(x) + \\ + 2(2\alpha+1)x(1-x^2)^\alpha h'(x) + (1-x^2)^{\alpha+1}h''(x) = 0. \end{aligned}$$

Now we choose $\alpha = \varepsilon/2$ and this equation becomes

$$(1-x^2)^{(\epsilon/2)+1}h''(x) - 2(\epsilon+1)x(1-x^2)^{\epsilon/2}h'(x) - (\lambda+\epsilon+\epsilon^2)(1-x^2)^{\epsilon/2}h(x) = 0 \quad .$$

Since $x \neq \pm 1$, we can divide by $(1-x^2)^{\epsilon/2}$. Then we have

$$(1-x^2)h''(x) - 2(\epsilon+1)xh'(x) - (\lambda+\epsilon+\epsilon^2)h(x) = 0$$

which can be rewritten as the new perturbed problem

$$(L+\epsilon B_1+\epsilon^2 B_2)h = \lambda h \quad (2.24)$$

where

$$Lh = ((1-x^2)h')', \quad B_1 h = -2xh' - h, \quad B_2 h = -h \quad . \quad (2.25)$$

We insert

$$h_\epsilon = \sum_{i=0}^{\infty} h_{in}(x)\epsilon^i$$

and

$$\lambda_\epsilon = \sum_{i=0}^{\infty} \lambda_{in}\epsilon^i$$

into equation (2.24) which yields:

$$(L-\lambda_{on})h_{on} = 0 \quad ,$$

$$(L-\lambda_{on})h_{in} = \lambda_{in}h_{on} - B_1 h_{on} \quad ,$$

$$(L-\lambda_{on})h_{2n} = \lambda_{1n}h_{1n} + \lambda_{2n}h_{on} - B_1 h_{in} - B_2 h_{on} \quad .$$

Again we impose the conditions

$$(h_{on}, h_{on}) = 1, (h_{on}, h_{jn}) = 0, \quad j = 1, 2, \dots$$

and take the inner product of each equation in the above system with h_{oi} , $i = 0, 1, \dots$. Hence,

$$\lambda_{1n} = (B_1 h_{on}, h_{on}), \quad (2.26)$$

$$h_{1n} = \sum_{\substack{i=0 \\ i \neq n}}^{\infty} \frac{h_{oi}}{\lambda_{on} - \lambda_{oi}} (B_1 h_{on}, h_{oi}), \quad (2.27)$$

$$\lambda_{2n} = (B_1 h_{1n}, h_{on}) + (B_2 h_{on}, h_{on}),$$

$$h_{2n} = \sum_{\substack{i=0 \\ i \neq n}}^{\infty} \frac{h_{oi}}{\lambda_{on} - \lambda_{oi}} [(B_1 h_{1n}, h_{oi}) + (B_2 h_{on}, h_{oi}) - \lambda_{1n} (h_{1n}, h_{oi})],$$

...

$$\lambda_{kn} = (B_1 h_{k-1,n}, h_{on}) + (B_2 h_{k-2,n}, h_{on}), \quad (2.28)$$

$$h_{kn} = \sum_{\substack{i=0 \\ i \neq n}}^{\infty} \frac{h_{io}}{\lambda_{on} - \lambda_{oi}} [(B_1 h_{k-1,n}, h_{oi}) + (B_2 h_{k-2,n}, h_{oi}) - \lambda_{1n} (h_{k-1,n}, h_{oi}) - \dots - \lambda_{k-1,n} (h_{1n}, h_{oi})]. \quad (2.29)$$

In order to proceed, we need the following result.

THEOREM 2.5. For non-negative integral values of n and i ,

$$\int_{-1}^1 x P_n'(x) P_i(x) dx = \begin{cases} 2 & \text{if } i < n \text{ and } n+i \text{ is odd,} \\ \frac{2n}{2n+1} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF: Using integration by parts, the case when $i = n$ follows immediately. If $i > n$, the result follows from Theorem 2.1. If $i < n$, we use integration by parts to get

$$\int_{-1}^1 x P_n'(x) P_i(x) dx = x P_i(x) P_n(x) \Big|_{-1}^1 - \int_{-1}^1 P_n(x) P_i'(x) dx - \int_{-1}^1 x P_i'(x) P_n(x) dx$$

which simplified to

$$\int_{-1}^1 x P_n'(x) P_i(x) dx = x P_i(x) P_n(x) \Big|_{-1}^1$$

by means of Theorem 2.1. Now from Lemma 2.1,

$$\int_{-1}^1 x P_n'(x) P_i(x) dx = \begin{cases} 2 & \text{if } i+n \text{ is odd,} \\ 0 & \text{if } i+n \text{ is even.} \end{cases}$$

This proves the theorem.

From (2.26) and the last theorem,

$$\lambda_{1n} = -2(xh'_{on}, h_{on})^{-1} = -(2n+1), \quad n = 0, 1, 2, \dots$$

Since $B_1 h = 2xh' - h$ we have

$$h_{1n} = \sum_{\substack{i=0 \\ i \neq n}}^{\infty} \frac{h_{oi}}{\lambda_{oi} - \lambda_{on}} 2(xh'_{on}, h_{oi}) = \sqrt{2(2n+1)} \sum_{\substack{i=0 \\ n+1 \leq i \\ i \text{ odd}}}^{n-1} \frac{(2i+1)P_i(x)}{\lambda_{oi} - \lambda_{on}}$$

from (2.27) and Theorem 2.5. Now

$$\lambda_{2n} = -2(xh'_{1n}, h_{on}) - 1 = 2(2n+1) \sum_{\substack{i=0 \\ n+i \text{ odd}}}^{n-1} \frac{(2i+1)}{\lambda_{on} - \lambda_{oi}} \int_{-1}^1 x P'_i(x) P_n(x) dx - 1$$

which simplifies to

$$\lambda_{2n} = -1, \quad n = 1, 2, \dots$$

because $i < n$. Substituting $\lambda_{1n} = -(2n+1)$ into the equation for h_{2n} we get

$$h_{2n} = \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{h_{ok}}{\lambda_{ok} - \lambda_{on}} [2(xh'_{1n}, h_{ok}) - 2n(h_{1n}, h_{ok})]$$

which can be written as

$$h_{2n} = \sqrt{2(2n+1)} \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{(2k+1)P_k(x)}{\lambda_{ok} - \lambda_{on}} \sum_{\substack{i=0 \\ n+i \text{ odd}}}^{n-1} \frac{2i+1}{\lambda_{oi} - \lambda_{on}} \times \\ \times \int_{-1}^1 x P'_i(x) P_k(x) dx - n \int_{-1}^1 P_i(x) P_k(x) dx$$

or

$$h_{2n} = \sqrt{2(2n+1)} \sum_{\substack{i=0 \\ n+i \text{ odd}}}^{n-1} \frac{2i+1}{\lambda_{oi} - \lambda_{on}} \frac{(i-n)P_i(x)}{\lambda_{oi} - \lambda_{on}} + \sum_{\substack{k=0 \\ k+i \text{ odd}}}^{i-1} \frac{(2k+1)P_k(x)}{\lambda_{ok} - \lambda_{on}}$$

by using Theorem 2.5. This simplifies nicely to

$$h_{2n} = \sqrt{2(2n+1)} \sum_{i=0}^{n-1} \frac{2i+1}{\lambda_{oi} - \lambda_{on}} Q_i^{(2)} P_i(x) \quad (2.30)$$

where

$$Q_i^{(2)} = \begin{cases} \frac{i-n}{\lambda_{oi}-\lambda_{on}} & \text{for } n+i \text{ odd,} \\ \sum_{\substack{\ell=i+1 \\ \ell+i \text{ odd}}}^{n-1} \frac{2\ell+1}{\lambda_{o\ell}-\lambda_{on}} & \text{for } n+i \text{ even.} \end{cases} \quad (2.31)$$

We let $k = 3$ in equation (2.28) to arrive at the following expression for λ_{3n} :

$$\lambda_{3n} = -2(xh'_{2n}, h_{on}) - (h_{2n}, h_{on}) - (h_{1n}, h_{on}) .$$

which simplifies immediately to

$$\lambda_{3n} = -2(xh'_{2n}, h_{on}) .$$

The degree of xh'_{2n} is less than n so by Theorem 2.1, $\lambda_{3n} = 0$, $n = 1, 2, \dots$. For all succeeding λ_{kn} 's the Legendre polynomial is always less than n and $\lambda_{kn} = 0$ for all $n = 0, 1, \dots$ and $k \geq 3$. Therefore we can write

$$\lambda_{\varepsilon} = \lambda_{on} + \lambda_{1n}\varepsilon + \lambda_{2n}\varepsilon^2 = -n(n+1) - (2n+1)\varepsilon - \varepsilon^2 . \quad (2.32)$$

Since $\lambda_{1n} = -(2n+1)$ and $\lambda_{2n} = -1$, when we let $k = 3$ in equation (2.29) we get

$$h_{3n} = \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{h_{ok}}{\lambda_{ok} - \lambda_{on}} [2(xh'_{2n}, h_{ok}) - 2n(h_{2n}, h_{ok})]$$

which can be rewritten as

$$h_{3n} = \sqrt{2(2n+1)} \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{(2k+1)P_k(x)}{\lambda_{ok} - \lambda_{on}} \sum_{i=0}^{n-1} \frac{(2i+1)Q_i^{(2)}}{\lambda_{oi} - \lambda_{on}} \times \\ \times \left[\int_{-1}^1 x P_i'(x) P_k(x) dx - n \int_{-1}^1 P_i(x) P_k(x) dx \right].$$

Now $\int_{-1}^1 x P_i'(x) P_k(x) dx$ and $\int_{-1}^1 P_i(x) P_k(x) dx$ are 0 if $k > i$.

Hence,

$$h_{3n} = \sqrt{2(2n+1)} \sum_{i=0}^{n-1} \frac{(2i+1)Q_i^{(2)}}{\lambda_{oi} - \lambda_{on}} \left[\frac{(2i-2n)P_i(x)}{\lambda_{oi} - \lambda_{on}} + \sum_{\substack{k=0 \\ k+i \text{ odd}}}^{i-1} \frac{(2k+1)P_k(x)}{\lambda_{ok} - \lambda_{on}} \right] \\ = 2\sqrt{2(2n+1)} \sum_{i=0}^{n-1} \frac{(2i+1)}{\lambda_{oi} - \lambda_{on}} \left[\frac{(i-n)P_i(x)}{\lambda_{oi} - \lambda_{on}} + \sum_{\substack{k=0 \\ k+i \text{ odd}}}^{i-1} \frac{(2k+1)P_k(x)}{\lambda_{ok} - \lambda_{on}} \right]$$

which is simply

$$h_{3n} = 2\sqrt{2(2n+1)} \sum_{i=0}^{n-1} \frac{2i+1}{\lambda_{oi} - \lambda_{on}} Q_i^{(3)} P_i(x) \quad (2.33)$$

where

$$Q_i^{(3)} = \begin{cases} \frac{i-n}{\lambda_{oi} - \lambda_{on}} Q_i^{(2)} & \text{for } i+n \text{ odd,} \\ \sum_{\substack{\ell=i+1 \\ \ell+i \text{ odd}}}^{n-1} \frac{2\ell+1}{\lambda_{o\ell} - \lambda_{on}} Q_i^{(2)} & \text{for } i+n \text{ even.} \end{cases} \quad (2.34)$$

Similarly,

$$h_{4n} = 2 \cdot 2\sqrt{2(2n+1)} \sum_{i=0}^{n-1} \frac{2i+1}{\lambda_{oi} - \lambda_{on}} Q_i^{(4)} P_i(x) \quad (2.35)$$

where

$$Q_i^{(4)} = \begin{cases} \frac{i-n}{\lambda_{oi} - \lambda_{on}} Q_i^{(3)} & \text{for } i+n \text{ odd,} \\ \sum_{\substack{\ell=i+1 \\ \ell+i \text{ odd}}}^{n-1} \frac{2\ell+1}{\lambda_{o\ell} - \lambda_{on}} Q_i^{(3)} & \text{for } i+n \text{ even.} \end{cases} \quad (2.36)$$

In general,

$$h_{kn} = 2^{k-2} \sqrt{2(2n+1)} \sum_{i=0}^{n-1} \frac{2i+1}{\lambda_{oi} - \lambda_{on}} Q_i^{(k)} P_i(x), \quad k = 2, 3, \dots \quad (2.37)$$

where

$$Q_i^{(k)} = \begin{cases} \frac{i-n}{\lambda_{oi} - \lambda_{on}} Q_i^{(k-1)} & \text{for } i+n \text{ odd,} \\ \sum_{\substack{\ell=i+1 \\ \ell+i \text{ odd}}}^{n-1} \frac{2\ell+1}{\lambda_{o\ell} - \lambda_{on}} Q_i^{(k-1)} & \text{for } i+n \text{ even.} \end{cases} \quad (2.38)$$

If $n+i$ is odd, $Q_i^{(2)}$ is bounded uniformly by $1/2$. However, if $n+i$ is even we get

$$Q_i^{(2)} = \sum_{\substack{\ell=i+1 \\ \ell+i \text{ odd}}}^{n-1} \frac{2\ell+1}{\lambda_{o\ell} - \lambda_{on}} = \sum_{\substack{\ell=i+1 \\ \ell+i \text{ odd}}}^{n-1} \left[\frac{1}{n-\ell} - \frac{1}{n+\ell+1} \right]$$

and $|Q_i^{(2)}| \leq n$ for all fixed n . Clearly,

$$|Q_i^{(k)}| \leq |Q_i^{(k-1)}| \cdot n^{k-2}$$

and $|Q_i^{(k)}| \leq n^{k-1}$. This yields the following:

$$\|h_{kn}\| \leq (2n)^k \sqrt{2n+1}$$

which implies that h_ε converges at least for $|\varepsilon| \leq \frac{1}{2n}$.

CHAPTER III

BESSEL'S DIFFERENTIAL EQUATION

Chapter III deals with Bessel's differential equation [10]. Once we have written the equation in self-adjoint form we use standard perturbation techniques [7] to obtain a system of equation [10]. Two examples, one with a general bounded linear perturbing operator and one with an unbounded linear perturbing operator, are studied.

1. THE ZEROS OF SOLUTIONS OF BESSEL'S EQUATION.

In physical applications Bessel's equation usually arises in the self-adjoint form

$$(xy')' + (k^2x - \frac{n^2}{x})y = 0, \quad x \neq 0, \quad (3.1)$$

with n and k real numbers. Its bounded solution for $n \geq 0$ is the Bessel function

$$J_n(kx) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(n+m+1)} \left(\frac{kx}{2}\right)^{2m+n}, \quad k \geq 0. \quad (3.2)$$

To ensure that these functions belong to $PC[0,1]$ we require $n \geq 0$. Otherwise, if n is not an integer, $J_n(kx)$ is unbounded as $x \rightarrow 0$, and is not piecewise continuous whereas if n is an integer, $J_{-n} = (-1)^n J_n$. Since we shall be working on the interval $[0,1]$ and since the leading coefficient of (3.1) vanishes at $x = 0$, we need

only impose the boundary condition $y(1) = 0$.

The following theorem is proved [10, p. 235] by applying Sturm's comparison theorem [10, p. 232] to

$$z'' + (k^2 + \frac{1-4n^2}{4x^2})z = 0 \quad (3.3)$$

which is obtained from (3.1) by making the change in variable $y = z/\sqrt{x}$.

THEOREM 3.1. The distance between successive zeros of every solution of Bessel's equation of order $n \geq 0$ is greater than π if $n > 1/2$, less than π if $0 \leq n < 1/2$ and equal to π if $n = 1/2$. Furthermore, this distance approaches π for all $n \geq 0$ as $x \rightarrow \infty$.

This result brings us to the next theorem which is of fundamental importance to our work in Sections 2 and 3 of this chapter.

THEOREM 3.2. Let $k_1 < k_2 < \dots$ be the positive zeros of $J_n(x)$ for $n \geq 0$. Then

$$\sum_{\substack{i=1 \\ i \neq m}}^{\infty} \frac{1}{|k_i^2 - k_m^2|} < \frac{1}{\sigma^2} \left(1 + \frac{3}{4m^2}\right)$$

where $\sigma = \min \{\pi, k_{i+1} - k_i\}$ for $i = 1, 2, \dots$.

PROOF: From Theorem 3.1, $k_1 > \sigma_1$. Hence if $i > m$,
 $k_i - k_m > \sigma(i-m)$ and

$$\frac{1}{k_i - k_m} < \frac{1}{\sigma(i-m)} , \quad \frac{1}{k_i + k_m} < \frac{1}{\sigma(i+m)} .$$

Therefore,

$$\frac{1}{k_i^2 - k_m^2} < \frac{1}{\sigma^2(i^2 - m^2)}.$$

Similarly, if $i < m$,

$$\frac{1}{k_m^2 - k_i^2} < \frac{1}{\sigma^2(m^2 - i^2)}$$

and

$$\begin{aligned} \sum_{\substack{i=1 \\ i \neq m}}^{\infty} \frac{1}{k_i^2 - k_m^2} &= \sum_{i=1}^{m-1} \frac{1}{k_m^2 - k_i^2} + \sum_{i=m+1}^{\infty} \frac{1}{k_i^2 - k_m^2} \\ &< \frac{1}{\sigma^2} \sum_{i=1}^{m-1} \frac{1}{m^2 - i^2} + \frac{1}{\sigma^2} \sum_{i=m+1}^{\infty} \frac{1}{i^2 - m^2}. \end{aligned}$$

Now [9, p. 74],

$$\sum_{\substack{i=1 \\ i \neq m}}^{\infty} \frac{1}{i^2 - m^2} = \sum_{i=1}^{m-1} \frac{1}{i^2 - m^2} + \sum_{i=m+1}^{\infty} \frac{1}{i^2 - m^2} = \frac{3}{4m^2}$$

which gives us

$$\sum_{\substack{i=1 \\ i \neq m}}^{\infty} \frac{1}{|k_i^2 - k_m^2|} < \frac{1}{\sigma^2} \left(1 + \frac{3}{4m^2}\right)$$

and the theorem is proved. We shall denote the right side of this inequality by $K(m)$.

The functions $\sqrt{x} J_n(k_j x)$, $j = 1, 2, \dots$, will be eigenvectors of the eigenvalue problem

$$Lz_j = \lambda_j z_j, \quad (3.4)$$

where $Lz_j = z_j'' + [(1-4n^2)/4x^2]z_j$ and $\lambda_j = -k_j^2$, if and only if $J_n(k_j) = 0$. Hence the positive zeros $k_1 < k_2 < \dots$ of J_n are eigenvalues and

$$\sqrt{x} J_n(k_j x), \quad j = 1, 2, \dots$$

are eigenvectors. Since x^n does not vanish when $x = 1$, $k_0 = 0$ is not an eigenvalue.

Since we will soon be computing series expansions relative to the eigenvector found above, we now evaluate their norms in $PC[0,1]$.

LEMMA 3.1. If k_j is the j^{th} positive zeros of J_n , $n \geq 0$,

$$\int_0^1 x J_n(k_i x) J_n(k_j x) dx = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{2} [J_{n+1}(k_j)]^2 & \text{if } i = j. \end{cases} \quad (3.5)$$

PROOF: We begin assuming $i \neq j$ and substitute $J_n(k_i x)$ and $J_n(k_j x)$ into (3.3):

$$\begin{aligned} [xJ_n'(k_i x)]' + (k_i^2 x - \frac{n^2}{x}) J_n(k_i x) &= 0, \\ [xJ_n'(k_j x)]' + (k_j^2 x - \frac{n^2}{x}) J_n(k_j x) &= 0. \end{aligned} \quad (3.6)$$

We multiply the first equation by $J_n(k_j x)$ and the second by $J_n(k_i x)$ and then subtract:

$$[J_n'(k_i x)]' J_n(k_j x) - [xJ_n'(k_j x)]' J_n(k_i x) + (k_i^2 - k_j^2) x J_n(k_i x) J_n(k_j x) = 0.$$

Now integrating from 0 to 1 and simplifying we get

$$\int_0^1 x J_n(k_i x) J_n(k_j x) dx = 0, \quad i \neq j.$$

To establish the second part of (3.4) we multiply (3.6) by $2xJ'_n(k_i x)$ to obtain

$$2x^2 J'_n J''_n + 2x(J'_n)^2 + 2(k_i^2 x^2 - n^2) J_n J'_n = 0$$

or

$$[x^2 (J'_n)^2]' + 2(k_i^2 x^2 - n^2) J_n J'_n = 0.$$

Rewriting this equation as

$$[x^2 (J'_n)^2]' + [(k_i^2 x^2 - n^2) J_n^2]' - 2k_i^2 x J_n^2 = 0$$

we get

$$2k_i^2 x [J_n(k_i x)]^2 = \frac{d}{dx} \{x^2 [J'_n(k_i x)]^2 + (k_i^2 x^2 - n^2) [J_n(k_i x)]^2\}.$$

Hence

$$\begin{aligned} 2k_i^2 \int_0^1 x [J_n(k_i x)]^2 dx &= x^2 [J'_n(k_i x)]^2 \Big|_0^1 + (k_i^2 x^2 - n^2) [J_n(k_i x)]^2 \Big|_0^1 \\ &= [J'_n(k_i)]^2. \end{aligned}$$

Therefore

$$\int_0^1 x [J_n(k_i x)]^2 dx = \frac{[J'_n(k_i)]^2}{2k_i^2}$$

which may be written with the aid of the well known recurrence relation

[10, p. 606]

$$xJ'_n - nJ_n = -xJ_{n+1}$$

as

$$\int_0^1 x [J_n(k_i x)]^2 dx = \frac{1}{2} [J_{n+1}(k_i)]^2 . \quad (3.7)$$

When rewritten in terms of the inner product on $PC[0,1]$, equation (3.7) becomes

$$||\sqrt{x} J_n(k_i x)||^2 = \frac{1}{2} [J_{n+1}(k_i)]^2 \quad (3.8)$$

and the eigenvectors $z_{oi} = C_i \sqrt{x} J_n(k_i x)$, where $C_i = \sqrt{2}/[J_{n+1}(k_i)]^2$, form an orthonormal set.

Now that we have established the existence of an infinite set of orthonormal functions for the boundary value problem (3.4) we need to determine whether these sets of functions are bases for $PC[0,1]$.

Let V_1 be the vector space of Bessel functions $y_j = J_n(k_j x)$, $j = 1, 2, \dots$, on the interval $[0,1]$ with the inner product $\langle y_j, y_i \rangle$ defined by

$$\langle y_j, y_i \rangle = \int_0^1 x y_j(x) y_i(x) dx .$$

Also, let V_2 be the vector space of functions $z_j = \sqrt{x} J_n(k_j x)$, $j = 1, 2, \dots$, on the interval $[0,1]$ where

$$(z_j, z_i) = \int_0^1 z_j(x) z_i(x) dx . \quad (3.9)$$

Now $z_j \in V_2$ if and only if $z_j/\sqrt{x} \in V_1$ which means

$$\left\langle \frac{z_j}{\sqrt{x}}, y_i \right\rangle = \int_0^1 \sqrt{x} z_j(x) y_i(x) dx = (z_j, z_i) \quad .$$

The following theorem [10, p. 621] is proved by Watson [17, pp. 591-593].

THEOREM 3.3. Let $k_1 < k_2 < \dots$ be the positive zeros of $J_n(x)$, and suppose $n \geq 0$. Then the functions $J_n(k_j x)$, $j = 1, 2, \dots$, are a basis for $PC[0,1]$, and every function in this space can be written uniquely in the form

$$f(x) = \sum_{j=1}^{\infty} C_j^2 \langle f, J_n(k_j x) \rangle J_n(k_j x)$$

where the series in question converges in the mean to f . Moreover, if f is piecewise smooth, this series converges uniformly on every closed subinterval of $(0,1)$ which does not contain a point of discontinuity of f .

From the discussion preceding Theorem 3.3 and since $z_{\ell i}$, $\ell = 1, 2, \dots$, satisfy a second order differential equation, we get the following corollary.

COROLLARY 3.1. Let $k_1 < k_2 < \dots$ be the positive zeros of $J_n(x)$ and suppose that $n \geq 1$. Then the functions $\sqrt{x} J_n(k_j x)$, $j = 1, 2, \dots$, are a basis for $PC[0,1]$ and every function in this space can be written in the form

$$z_{li} = \sum_{j=1}^{\infty} (z_{li}, z_{oj}) z_{oj}$$

where the series in question converges uniformly to z_{li} on $(0,1)$ and (z_{li}, z_{oj}) is given by (3.9).

2. PERTURBATION OF BESSEL'S EQUATION BY A BOUNDED OPERATOR.

As was mentioned in Chapter I, if the perturbing operator B is a bounded linear operator then $\|Bz\| \leq \|B\| \|z\|$ for all functions z . From (1.14), $\lambda_{1j} = (Bz_{oj}, z_{oj})$ and

$$|\lambda_{1j}| \leq |(Bz_{oj}, z_{oj})| \leq \|B\|.$$

by the Schwarz inequality [10, p. 263]. Similarly, $|\lambda_{2j}| \leq \|B\|, \dots$, so the series (1.8) converges for all $|\varepsilon| \leq 1$.

By Theorem 3.3 we can write

$$z_{1j} = \sum_{i=1}^{\infty} (z_{1j}, z_{oi}) z_{oi}.$$

The inner product (z_{1j}, z_{oi}) is derived from the second equation in (1.11) as follows:

$$(L - \lambda_{oj}) z_{1j} = \lambda_{1j} z_{oj} - Bz_{oj},$$

$$(Lz_{1j}, z_{oi}) - \lambda_{oj} (z_{1j}, z_{oi}) = \lambda_{1j} (z_{1j}, z_{oi}) - (Bz_{oj}, z_{oi}), \quad i \neq j.$$

Now $(Lz_{1j}, z_{oi}) = (z_{1j}, Lz_{oi}) = \lambda_{oi} (z_{1j}, z_{oi})$ from (1.7) and the above equation becomes

$$(\lambda_{oi} - \lambda_{oj})(z_{1j}, z_{oi}) = \lambda_{1j}(z_{oj}, z_{oi}) - (Bz_{oj}, z_{oi}) .$$

Hence

$$(z_{1j}, z_{oi}) = \frac{(Bz_{oj}, z_{oi})}{\lambda_{oj} - \lambda_{oi}} ,$$

and

$$z_{1j} = \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \frac{(Bz_{oj}, z_{oi}) z_{oi}}{\lambda_{oj} - \lambda_{oi}} . \quad (3.10)$$

Therefore

$$\|z_{1j}\| \leq \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \frac{\|B\|}{|k_i^2 - k_j^2|} \leq \|B\| \cdot K(j) .$$

Similarly,

$$z_{2j} = \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \frac{z_{oi}}{\lambda_{oj} - \lambda_{oi}} [(Bz_{1j}, z_{oi}) - \lambda_{1j}(z_{1j}, z_{oi})]$$

which implies

$$\|z_{2j}\| \leq \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \frac{\|B\| + |\lambda_{1j}| \|z_{1j}\|}{|k_i^2 - k_j^2|} \leq \|B\| \cdot K(j) + \|B\|^2 \cdot K^2(j) .$$

For the third equation in (1.11) we get

$$z_{3j} = \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \frac{z_{oi}}{\lambda_{oj} - \lambda_{oi}} [(Bz_{2j}, z_{oi}) - \lambda_{1j}(z_{2j}, z_{oi}) - \lambda_{2j}(z_{1j}, z_{oi})^2] \quad (3.11)$$

and

$$\|z_{3j}\| \leq \|B\| \cdot K(j) - 2\|B\|^2 K^2(j) + \|B\|^3 K^3(j) \quad .$$

In general,

$$z_{\ell j} = \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \frac{z_{oi}}{\lambda_{oj} - \lambda_{oi}} \left[(Bz_{\ell-1,j}, z_{oi})^{-\lambda_{1j}} (z_{\ell-1,j}, z_{oi}) - \dots - \lambda_{\ell-1,j} (z_{1j}, z_{oi}) \right] \quad (3.12)$$

and

$$\|z_{\ell j}\| \leq \sum_{i=0}^{\ell-1} \binom{\ell-1}{i} [K(j)\|B\|]^{i+1} = K(j)\|B\| [1+K(j)\|B\|]^{\ell-1} \quad .$$

Therefore (1.9) converges at least for

$$|\varepsilon| < \frac{1}{1 + K(j)\|B\|}$$

3. AN EXAMPLE WITH A FIRST ORDER DIFFERENTIAL PERTURBING OPERATOR.

We begin by defining B by the relation

$$By = \sum_{i=0}^{\infty} \frac{(g_1, z_{oi}) z'_{oi}}{k_i} \quad , \quad (' = \frac{d}{dx}) \quad (3.13)$$

for a function $g \in C^2[0,1]$ where

$$g(x) = \sum_{i=0}^{\infty} (g, z_{oi}) z_{oi} \quad . \quad (3.14)$$

To verify that B given by (3.13) is a well defined operator we require

the following theorem.

THEOREM 3.4. Let k_i be the i^{th} zero of J_n . Then for all $n \geq 1/2$,

$$\frac{\|z'_{oi}\|}{k_i} \leq 1. \quad (3.15)$$

PROOF: By definition

$$\begin{aligned} \frac{\|z'_{oi}\|}{k_i} &= \frac{1}{k_i} \left\{ \int_0^1 [z'_{oi}(x)]^2 dx \right\}^{1/2} \\ &= \frac{1}{k_i} \left\{ z'_{oi}(x) z_{oi}(x) \Big|_0^1 - \int_0^1 z'_{oi}(x) z''_{oi}(x) dx \right\}^{1/2}. \end{aligned} \quad (3.16)$$

Now from (3.3)

$$z''_{oi} = \left(\frac{4n^2 - 1}{4x^2} - k_i^2 \right) z_{oi}$$

and if we substitute for z''_{oi} in (3.16) we get

$$\frac{\|z''_{oi}\|}{k_i} = \left\{ 1 + \frac{1-4n^2}{4k_i^2} \int_0^1 \left[\frac{z_{oi}(x)}{x} \right]^2 dx \right\}^{1/2}$$

which is always positive. Since

$$\frac{1-4n^2}{4k_i^2} \int_0^1 \left[\frac{z_{oi}(x)}{x} \right]^2 dx$$

is negative for $n \geq 1/2$, (3.15) follows.

Now using (3.15) we get

$$\|Bg(x)\| \leq \sum_{i=0}^{\infty} |(g, z_{oi})| \frac{\|z'_{oi}\|}{k_i} \leq \sum_{i=0}^{\infty} |(g, z_{oi})|$$

and from (3.14), this is bounded. Therefore B is a well defined operator which is clearly linear.

Therefore

$$Bz_{oj} = \sum_{i=0}^{\infty} \frac{(z_{oj}, z_{oi}) z'_{oi}}{k_i} = \frac{z'_{oj}}{k_j} \quad (3.17)$$

which implies

$$|\lambda_{1j}| \leq \|Bz_{oj}\| \leq 1 \quad (3.18)$$

From (3.10),

$$z_{1j} = \sum_{\substack{\ell=0 \\ \ell \neq j}}^{\infty} \frac{(Bz_{oj}, z_{o\ell}) z_{o\ell}}{k_{\ell}^2 - k_j^2} = \sum_{\substack{\ell=1 \\ \ell \neq j}}^{\infty} \frac{z_{o\ell}}{k_{\ell}^2 - k_j^2} (Bz_{oj}, z_{o\ell})$$

Therefore

$$\|z_{1j}\| < \sum_{\substack{\ell=0 \\ \ell \neq j}}^{\infty} \frac{\|Bz_{oj}\|}{|k_{\ell}^2 - k_j^2|} < \sum_{\substack{\ell=0 \\ \ell \neq j}}^{\infty} \frac{1}{|k_{\ell}^2 - k_j^2|} \quad (3.19)$$

and from Theorem 3.2, $\|z_{1j}\| \leq K(j)$.

Since $\lambda_{2j} = (Bz_{1j}, z_{oj})$ we get

$$|\lambda_{2j}| \leq \|Bz_{1j}\|$$

and we must find a bound for $\|Bz_{1j}\|$. We know

$$\begin{aligned} Bz_{1j} &= \sum_{i=0}^{\infty} \frac{z'_{oi}}{k_i} \sum_{\substack{\ell=0 \\ \ell \neq j}}^{\infty} \frac{(Bz_{oj}, z_{o\ell})}{k_{\ell}^2 - k_j^2} (z_{o\ell}, z_{oi}) \\ &= \sum_{\substack{i=0 \\ i \neq j}}^{\infty} \frac{Bz_{oi} (Bz_{oj}, z_{oi})}{k_i^2 - k_j^2} \end{aligned}$$

and

$$|\lambda_{2j}| \leq \|Bz_{1j}\| \leq \sum_{\substack{i=0 \\ i \neq j}}^{\infty} \frac{\|Bz_{oi}\| \cdot \|Bz_{oj}\|}{|k_i^2 - k_j^2|} \leq K(j) \quad (3.20)$$

From (3.10),

$$z_{2j} = \sum_{\substack{m=0 \\ m \neq j}}^{\infty} (z_{2j}, z_{om}) z_{om} = \sum_{\substack{m=0 \\ m \neq j}}^{\infty} \frac{z_{om}}{\lambda_{oj} - \lambda_{om}} [(Bz_{1j}, z_{om}) - \lambda_{1j} (z_{1j}, z_{om})]$$

which can be bounded as follows:

$$\|z_{2j}\| \leq \sum_{\substack{m=0 \\ m \neq j}}^{\infty} \frac{1}{|k_m^2 - k_j^2|} [\|Bz_{1j}\| + |\lambda_{1j}| \|z_{1j}\|] \leq 2K^2(j) \quad (3.21)$$

Again:

$$Bz_{2j} = \sum_{i=0}^{\infty} \frac{(z_{2j}, z_{oi}) z'_{oi}}{k_i} = \sum_{\substack{i=0 \\ i \neq j}}^{\infty} \frac{Bz_{oi}}{\lambda_{oj} - \lambda_{oi}} [(Bz_{1j}, z_{oi}) - \lambda_{1j} (z_{1j}, z_{oi})]$$

and

$$|\lambda_{3j}| \leq \|Bz_{2j}\| \leq \sum_{\substack{i=1 \\ i \neq j}}^{\infty} \frac{\|Bz_{oi}\|}{|k_i^2 - k_j^2|} [\|Bz_{1j}\| + |\lambda_{1j}| \|z_{1j}\|] \leq 2K^2(j) .$$

Hence

$$\|z_{3j}\| < \sum_{\substack{p=1 \\ p \neq j}}^{\infty} \frac{\|z_{op}\|}{|k_p^2 - k_j^2|} [\|Bz_{2j}\| + |\lambda_{1j}| \|z_{2j}\| + |\lambda_{2j}| \|z_{1j}\|] \leq 5K^3(j) .$$

In general,

$$|\lambda_{\ell j}| \leq a_{\ell-1}[K(j)]^{-1}, \quad \ell = 1, 2, \dots \quad (3.22)$$

and

$$\|z_{\ell j}\| \leq a_{\ell}[K(j)]^{\ell}, \quad \ell = 1, 2, \dots \quad (3.23)$$

where $K(j)$ is given by Theorem 3.2 and

$$a_0 = 1, a_1 = 1, a_{\ell} = \sum_{i=0}^{\ell-1} a_i a_{\ell-i-1}, \quad \ell = 2, 3, \dots \quad (3.24)$$

We are now in a position to consider the convergence of

$$\lambda_{\epsilon} = \lambda_{oj} + \sum_{\ell=1}^{\infty} \lambda_{\ell j} \epsilon^{\ell}, \quad z_{\epsilon} = z_{oj} + \sum_{\ell=1}^{\infty} z_{\ell j} \epsilon^{\ell} . \quad (3.25)$$

Let us assume that a function $f(x)$ converges for some values of x ,
to the sum

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad (3.26)$$

where

$$a_k = \sum_{i=0}^{k-1} a_i a_{k-i-1} \quad .$$

Then

$$f^2(x) = \sum_{k=0}^{\infty} C_k x^k$$

where $C_k = a_{k+1}$ and we get the equation $xf^2(x) - f(x) + 1 = 0$,
 which has two branches; only one of which is a continuous relation.
 Solving for f on this branch,

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

which enables us to assert that f is a continuous function for all
 $-\infty < x \leq 1/4$ and $0 < f(x) \leq 2$. Now (3.26) converges for all
 $|x| \leq 1/4$ and the series (3.25) converge at least for

$$|\varepsilon| \leq \frac{1}{4K(j)} \quad .$$

CHAPTER IV

CONCLUSION

Perturbation of eigenvalue problems has been thoroughly studied, particularly for bounded perturbing operators, by Franz Rellich [12], [13]. His fundamental results [13] have not been generalized for the unbounded case. The application of these techniques to Legendre's second order differential operator [10] and Bessel's second order differential operator [10] has not been found in the literature.

When we finally set these problems up correctly, we were left with the difficult task of choosing perturbing operators that would yield meaningful results. Rellich shows that a small parameter ϵ does not necessarily imply a small perturbation [12].

With respect to Legendre's differential equation, we began by considering the equation

$$(L+\epsilon B)y_{\epsilon} = \lambda_{\epsilon} y_{\epsilon} \quad (4.1)$$

where L is Legendre's differential operator and $By_{\epsilon} = -\lambda_{\epsilon} y_{\epsilon}$. The anticipated, relatively uninteresting results were obtained - the eigenvectors were not perturbed but the eigenvalues were. Finally we perturbed (4.1) with the perturbing operators $By = y'$ and $By = (B_1 + \epsilon B_2)y$ where $B_1 y = -2xy'_1 - y_1$ and $B_2 y = -y$. Fortunately, the coefficients in the series expansions of y_{ϵ} and λ_{ϵ} reduced to

finite sums and could be easily bounded.

Several attempts at perturbing Bessel's equation were unsuccessful. We considered

$$(xy')' + \frac{k^2 y}{x} = \frac{\lambda y}{x} \quad (4.2)$$

where $\lambda = n^2$, $n = 0, 1, \dots$, and k is a zero of J_n . Most of the expressions for this problem contained x^{-1} and were not explicitly integrable. We were unable to find any estimates on regions of convergence for this case.

The transformation $y = z/\sqrt{x}$ gave us a nice boundary value problem to perturb. First of all, we considered a bounded perturbing operator to demonstrate Rellich's result that a convergent series exists for the eigenvalues and eigenvectors of the perturbed problem. Then we let $By = y'$. A uniform bound on Bz_{oj} , where $z_{oj} = C_j \sqrt{x} J_n(k_j x)$, was not found because

$$\|Bz_{oj}\| = \left\{ \int_0^1 [z'_{oj}(x)]^2 dx \right\}^{1/2} = \left\{ k_j^2 + \left(\frac{1}{4} - n^2\right) \int_0^1 \left[\frac{z_{oj}(x)}{x} \right]^2 dx \right\} > 0.$$

Since $(\frac{1}{4} - n^2) \int_0^1 [z_{oj}^2/x^2] dx \leq 0$ for $n \geq \frac{1}{2}$, $\|Bz_{oj}\| \leq k_j$ which is not a uniform bound. Eventually, the operator

$$Bz = \sum_{i=0}^{\infty} \frac{(z, z_{oi}) z'_{oi}}{k_i}$$

produced convergent series solution for Bessel's perturbed equation.

In this thesis, we have chosen the perturbing operators B with no obvious physical importance. However, the fact that first order differential perturbing operators yield well defined eigenvalues and eigenvectors for the perturbed eigenvalue problem is significant.

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